Homotopy algebras and colour-kinematics duality

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Poincaré doesn't meddle with Lie

- We consider Yang–Mills (YM) theory

$$S = -\frac{1}{4}\int \mathrm{d}^d x\, F^{a\mu\nu} F^a_{\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{bc}{}^a A^b_\mu A^c_\nu$$

- A^a_μ has two indeces with a very different meaning: gauge index a (internal symmetry), Lorentz index μ (spacetime symmetry)

- Coleman–Mandula theorem says that is impossible to combine internal symmetries and spacetime symmetries in any but a trivial way

Poincaré doesn't meddle with Lie

What is the most general symmetry algebra $\mathfrak{S}$ of a QFT that leaves its S matrix invariant?

Coleman–Mandula theorem (1967)

If we consider a theory with:

- $\mathfrak{S}$ containing Poincaré algebra $\mathfrak{p}$
- finite number of particles with mass less than M, for every $M > 0$
- nontrivial S matrix that is an analytic function of s and t
- *other technical assumptions*

then $\mathfrak{S} = \mathfrak{p} \oplus \mathfrak{g}$, with $\mathfrak{g}$ a Lie algebra

Flash review of CK duality and double copy

- In general, we can write n-points L-loops YM amplitude as sums of trivalent graphs

$$\mathcal{A}_{n,L}^{\mathrm{YM}} = \sum_i \int \prod_{l=1}^{L} \mathrm{d}^d p_l \frac{1}{S_i} \frac{C_i N_i}{D_i}$$

- i ranges over all trivalent L-loops graphs
- C_i: colour factor, composed of gauge group structure constants
- N_i: kinematic factor, composed of Lorentz-invariant contractions of polarisations and momenta

Flash review of CK duality and double copy

- Generalised gauge transformation

$$N_i \;\mapsto\; N_i + \Delta_i, \qquad \sum_i \int \prod_{l=1}^{L} \mathrm{d}^d p_l \frac{1}{S_i} \frac{C_i \Delta_i}{D_i} \;=\; 0$$

Bern–Carrasco–Johansson colour–kinematics duality (2008)

There is a choice of kinematic factors such that N_is obey the same algebraic relations (e.g., Jacobi identity) of the correspondent C_i

- True at tree-level, conjectured for loop-level
- If true, it would allow us to compute gravity amplitudes from YM ones

Flash review of CK duality and double copy

Yang–Mills double copy

If CK duality holds true, replacing the colour factor with a copy of the kinematic factor in $\mathcal{A}^{\text{YM}}_{n,L}$ produces a $\mathcal{N}=0$ supergravity amplitude

$$\mathcal{A}^{\text{YM}}_{n,L} = \sum_i \int \prod_{l=1}^{L} \mathrm{d}^d p_l \frac{1}{S_i} \frac{C_i N_i}{D_i} \rightarrow \mathcal{A}^{\mathcal{N}=0}_{n,L} = \sum_i \int \prod_{l=1}^{L} \mathrm{d}^d p_l \frac{1}{S_i} \frac{\tilde{N}_i N_i}{D_i}$$

- All-loop statement, the problem is then to validate CK duality at loop level
- Until now, on-shell scattering amplitude approach: an off-shell Lagrangian realisation of colour-kinematics duality and double copy could solve the all-loop conundrum ☞ Leron's talk
- Homotopy algebras provide a natural setting for colour–kinematics factorisation and Lagrangian double copy

Homotopy algebras

- Informally, homotopy algebras are generalizations of classical algebras (e.g., associative, Lie) where the respective structural identities (e.g., associativity, Jacobi identity) hold up to homotopies

Classical algebra	Homotopy algebra
Associative algebra	A_∞-algebra
Associative commutative algebra	C_∞-algebra
Lie algebra	L_∞-algebra

- Homotopy structures are ubiquitous in Physics: while homotopy algebras emerged in the context of string field theory, they were later recognized as underlying structures of every Lagrangian field theory

BV formalism: motivations

- Batalin–Vilkovisky (BV) formalism is the bridge between (quantum) field theories and homotopy algebras
- To quantize a classical theory means to make sense of the path integral

$$\int_{\mathfrak{F}} \mu_{\mathfrak{F}}(\Phi) \; e^{\frac{i}{\hbar} S[\Phi]}$$

- Standard approach: BRST formalism
- If the symmetries close off-shell, then BRST formalism is enough for quantization

BV formalism

- In the case of open symmetries, BRST complex is a complex only up to e.o.m.
- The BV quantisation is a sophisticated machinery, that allows us to gauge-fix and quantize these complicated field theories
- We extend the BRST complex, doubling the field content of the theory

$$\mathfrak{F}_{\mathrm{BV}} \;=\; T^*[1]\mathfrak{F}_{\mathrm{BRST}}$$

- Fields Φ^A are local coordinates on $\mathfrak{F}_{\mathrm{BRST}}$, antifields Φ_A^+ are fibre coordinates. As a cotangent bundle, $\mathfrak{F}_{\mathrm{BV}}$ comes with a natural symplectic structure and Poisson brackets $\{-,-\}$

BV formalism

- We extend Q_{BRST} and S_{BRST} to Q_{BV} and S_{BV}, requiring

$$Q_{\text{BV}}|_{\mathfrak{F}_{\text{BRST}}} = Q_{\text{BRST}}, \quad Q_{\text{BV}} = \{S_{\text{BV}}, -\}$$

$$Q_{\text{BV}} S_{\text{BV}} = \{S_{\text{BV}}, S_{\text{BV}}\} = 0$$

  the latter is known as *BV master equation*

- The differential algebra of the BV formalism dualizes to a codifferential coalgebra, that can be equivalently described as an L_∞-algebra

BV formalism: gauge-fixing and quantization

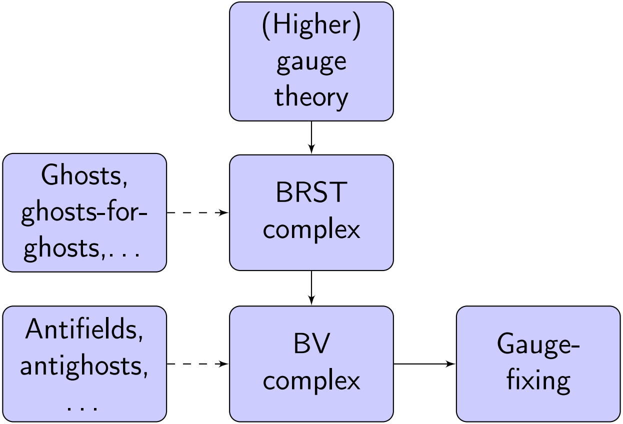
- Before quantization: imposing gauge-fixing in the BV formalism
- Gauge-fixing S_{BV} means evaluating it on an appropriate Lagrangian submanifold of $\mathfrak{F}_{\mathrm{BV}}$
- We eliminate the antifields by introducing a gauge-fixing fermion Ψ:

$$\Phi_A^+ = \frac{\delta}{\delta \Phi^A}\Psi$$

- Gauge-independence of the expectation values for observables: BV quantum master equation

$$\{S_{\mathrm{BV}}^{\hbar}, S_{\mathrm{BV}}^{\hbar}\} - 2i\hbar\Delta_{\mathrm{BV}} S_{\mathrm{BV}}^{\hbar} = 0$$

BV formalism: summary



The dual picture: L_∞-algebras

- BRST and BV formalism introduce a differential Q on the graded commutative algebra of polynomial functions of fields, $\mathscr{C}^\infty(\mathfrak{F}[1])$
- This is an instance of an abstract geometrical construction, Q-vector spaces
- In the simplest case, we have an ordinary vector space $\mathfrak{g}$ with basis e^a, and the most general degree 1 differential acting on $\mathscr{C}^\infty(\mathfrak{g}[1])$ is

$$Q\xi^a = -\frac{1}{2}\mathrm{f}^a_{bc}\xi^b\xi^c,$$

where the coordinate functions ξ^a are basis for $\mathfrak{g}^*$
- Requiring $Q^2 = 0$ is equivalent to require Jacobi identity for f^a_{bc}, i.e. that f^a_{bc} are the structure constant of a Lie algebra with bracket $[\mathrm{e}_b, \mathrm{e}_c] = \mathrm{f}^a_{bc}\mathrm{e}_a$

- The differential algebra picture and the Lie algebra picture are easy to relate, introducing contracted coordinate functions

$$\mathsf{a} \;=\; \xi^a \otimes \mathsf{e}_a \;\in\; (\mathfrak{g}[1])^* \otimes \mathfrak{g}$$

$$Q\mathsf{a} \;=\; (Q\xi^a)\otimes \mathsf{e}_a \;=\; -\frac{1}{2}\mathsf{f}^a_{bc}\xi^b\xi^c\otimes \mathsf{e}_a \;=\; -\frac{1}{2}\xi^b\xi^c\otimes[\mathsf{e}_b,\mathsf{e}_c] \;=\; -\frac{1}{2}[\mathsf{a},\mathsf{a}]$$

- More general vector fields: we consider now the graded vector space $\mathfrak{F}_{\mathrm{BV}}$

$$\mathsf{a} \;=\; \Phi^I \otimes \mathsf{e}_I + \Phi^+_I \otimes \mathsf{e}^I \;\in\; (\mathfrak{F}_{\mathrm{BV}}[1])^* \otimes \mathfrak{F}_{\mathrm{BV}}$$

$$Q\mathsf{a} \;=\; -\sum_i \frac{1}{i!}\mu_i(\mathsf{a},\ldots,\mathsf{a})$$

- The multibrackets are multilinear, graded antisymmetric maps, called *higher products*

L_∞-algebras

$$Qa = -\sum_i \frac{1}{i!}\mu_i(a, \dots, a)$$

- Requiring $Q^2 = 0$ is equivalent to require that $(\mathfrak{F}_{\mathrm{BV}}, \mu_i)$ is an L_∞-algebra
- An L_∞-algebra is a graded vector space equipped with higher products, that satisfy a generalization of Jacobi identity
- Underlying every Lagrangian field theory is an L_∞-algebra that encodes the whole classical theory (symmetries, fields, equations of motion, Noether identities. . .)

- Let $\mathfrak{L} := \bigoplus_{k\in\mathbb{Z}} \mathfrak{L}_k$ be a $\mathbb{Z}$-graded vector space. Elements of $\mathfrak{L}_k$ are said to be homogeneous and of degree k, and we shall denote the degree of a homogeneous element $\ell \in \mathfrak{L}$ by $|\ell| \in \mathbb{Z}$.

- Suppose there is a differential $\mu_1 : \mathfrak{L} \to \mathfrak{L}$ of degree 1. This allows us to consider the chain complex

$$\cdots \xrightarrow{\mu_1} \mathfrak{L}_{-1} \xrightarrow{\mu_1} \mathfrak{L}_0 \xrightarrow{\mu_1} \mathfrak{L}_1 \xrightarrow{\mu_1} \cdots$$

- We equip this complex with products $\mu_i : \mathfrak{L} \times \cdots \times \mathfrak{L} \to \mathfrak{L}$ of degree $2 - i$ for $i \in \mathbb{N}$, which are i-linear, graded antisymmetric and subject to the *homotopy Jacobi identity*

$$\sum_{i_1+i_2=i} \sum_{\sigma \in \mathrm{Sh}(i_1;i)} \pm \mu_{i_2+1}(\mu_{i_1}(\ell_{\sigma(1)}, \dots, \ell_{\sigma(i_1)}), \ell_{\sigma(i_1+1)}, \dots, \ell_{\sigma(i)}) = 0$$

- The first three identities are:

$$\mu_1(\mu_1(\ell_1)) = 0 ,$$

$$\mu_1(\mu_2(\ell_1, \ell_2)) = \mu_2(\mu_1(\ell_1), \ell_2) \pm \mu_2(\ell_1, \mu_1(\ell_2)) ,$$

$$\begin{aligned} \mu_2(\mu_2(\ell_1, \ell_2), \ell_3) \pm \mu_2(\mu_2(\ell_2, \ell_3), \ell_1) \pm \mu_2(\mu_2(\ell_3, \ell_1), \ell_2) = \\ = \mu_1(\mu_3(\ell_1, \ell_2, \ell_3)) + \mu_3(\mu_1(\ell_1), \ell_2, \ell_3) + \\ \pm \mu_3(\ell_1, \mu_1(\ell_2), \ell_3) \pm \mu_3(\ell_1, \ell_2, \mu_1(\ell_3)) \end{aligned}$$

- We call $(\mathfrak{L}, \mu_i)$ an L_∞-algebra

- A graded vector space $\mathfrak{A}$, with degree $2-i$ i-linear higher products $m_i : \mathfrak{A} \times \cdots \times \mathfrak{A} \to \mathfrak{A}$, subject to the *homotopy associativity relations*

$$m_1(m_1(\ell_1)) \;=\; 0 \;,$$

$$m_1(m_2(\ell_1,\ell_2)) = m_2(m_1(\ell_1),\ell_2) \pm m_2(\ell_1, m_1(\ell_2)) \;,$$

$$m_1(m_3(\ell_1,\ell_2,\ell_3)) + m_3(m_1(\ell_1),\ell_2,\ell_3) + \pm m_3(\ell_1, m_1(\ell_2),\ell_3) +$$
$$\pm m_3(\ell_1,\ell_2,m_1(\ell_3)) = m_2(m_2(\ell_1,\ell_2),\ell_3) + m_2(\ell_1, m_2(\ell_2,\ell_3))$$
$$\cdots$$

  We call $(\mathfrak{A}, m_i)$ an A_∞-algebra

- If in addition m_i satisfy *homotopy commutativity relations*

$$m_2(\ell_1,\ell_2) = \pm m_2(\ell_2,\ell_1)$$
$$m_3(\ell_1,\ell_2,\ell_3) \pm m_3(\ell_1,\ell_3,\ell_2) \pm m_3(\ell_3,\ell_1,\ell_2) = 0$$
$$\cdots$$

  $(\mathfrak{A}, m_i)$ is a C_∞-algebra

Morphisms of L_∞-algebras

A morphism of L_∞-algebras is a collection ϕ of i-linear totally graded antisymmetric maps $\phi_i : \mathfrak{L} \times \cdots \times \mathfrak{L} \to \mathfrak{L}'$ of degree $1-i$ such that

$$\sum_{j+k=i} \sum_{\sigma \in \bar{\mathrm{Sh}}(j;i)} (\pm 1) \times \phi_{k+1}(\mu_j(\ell_{\sigma(1)}, \dots, \ell_{\sigma(j)}), \ell_{\sigma(j+1)}, \dots, \ell_{\sigma(i)})$$

$$= \sum_{j=1}^{i} \frac{1}{j!} \sum_{k_1+\cdots+k_j=i} \sum_{\sigma \in \bar{\mathrm{Sh}}(k_1,\dots,k_{j-1};i)} (\pm 1) \times$$

$$\times \mu'_j\Big(\phi_{k_1}\big(\ell_{\sigma(1)}, \dots, \ell_{\sigma(k_1)}\big), \dots, \phi_{k_j}\big(\ell_{\sigma(k_1+\cdots+k_{j-1}+1)}, \dots, \ell_{\sigma(i)}\big)\Big)$$

If ϕ_1 is invertible, then ϕ is called an *isomorphism*. If ϕ_1 induces an isomorphism of cohomology rings $H^\bullet_{\mu_1}(\mathfrak{L}) \cong H^\bullet_{\mu_1}(\mathfrak{L}')$, then ϕ is called a *quasi-isomorphism*

Cyclic structure

We say that $\mathfrak{L}$ is a *cyclic* L_∞-algebra if it is endowed with a non-degenerate bilinear graded symmetric pairing $\langle -, - \rangle : \mathfrak{L} \times \mathfrak{L} \to \mathbb{R}$ of degree k which is cyclic in the sense of

$$\langle \ell_1, \mu_i(\ell_2, \ldots, \ell_{i+1}) \rangle \; = \; \pm \langle \ell_{i+1}, \mu_i(\ell_1, \ldots, \ell_i) \rangle$$

Example

- Vector space: $\mathfrak{L} = \mathfrak{gl}(n, \mathbb{C})$
- Higher products: $\mu_i = 0$ for $i \neq 2$, $\mu_2(\ell_1, \ell_2) = [\ell_1, \ell_2]$
- Cyclic structure: $\langle \ell_1, \ell_2 \rangle = \mathrm{Tr}(\ell_1^\dagger \ell_2)$

Maurer–Cartan (MC) homotopy theory

- Given an L_∞-algebra $(\mathfrak{L}, m_i)$, we call an element of degree 1, $a \in \mathfrak{L}_1$, a *gauge potential*
- We define the *curvature* of a by

$$f \;=\; \sum_{i\geq 1} \frac{1}{i!}\mu_i(a, \ldots, a) \;\in\; \mathfrak{L}_2.$$

- The curvature f obeys the *Bianchi identity*

$$\sum_{i\geq 0} \pm\frac{1}{i!}\mu_{i+1}(f, a \ldots, a) \;=\; 0$$

Maurer–Cartan (MC) homotopy theory

- a is a MC element if it satisfies the MC equation $f = 0$
- The MC equation is variational whenever $(\mathfrak{L}, \mu_i, \langle -, - \rangle)$ is a cyclic L_∞-algebra. The MC action is given by

$$S = \sum_{i \geq 1} \frac{1}{(i+1)!} \langle a, \mu_i(a, \ldots, a) \rangle$$

- Infinitesimal gauge transformations are mediated by degree 0 elements $c_0 \in \mathfrak{L}_0$

$$\delta_{c_0} a = \sum_{i \geq 0} \frac{1}{i!} \mu_{i+1}(a, \ldots, a, c_0)$$

- The gauge parameters $c_0 \in \mathfrak{L}_0$ may enjoy gauge freedom themselves which is mediated by *next-to-lowest* gauge parameters $c_{-1} \in \mathfrak{L}_{-1}$ of degree -1, and so on

Yang–Mills theory in the BV formalism

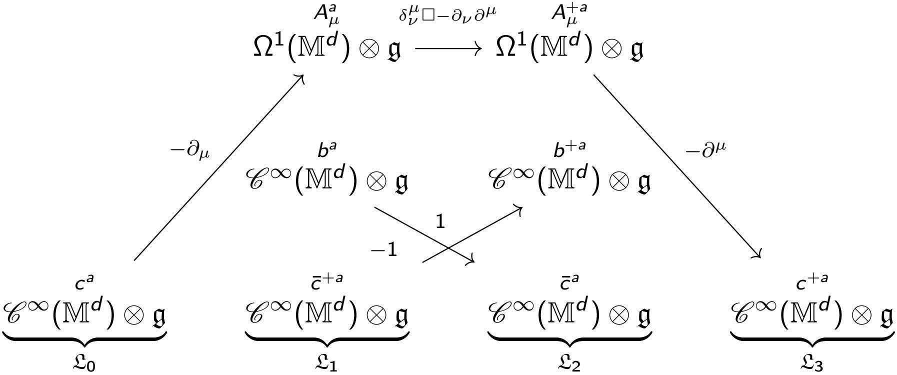
- Extend YM action with antifields (A^+, c^+) and trivial pairs (b, $\bar{c}^+$, b^+, $\bar{c}$)

$$S_{\text{BV}}^{\text{YM}} = \int_{\mathbb{M}^d} \mathrm{d}^d x \left\{ -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + A^{+a}_{\mu} (\nabla^\mu c)^a + \tfrac{g}{2} f^a_{bc} c^{+a} c^b c^c + b^a \bar{c}^{+a} \right\}$$

- We can formulate YM theory as the Maurer–Cartan homotopy theory associated to a cyclic L_∞-algebra $(\mathfrak{L}, \mu_i, \langle -, - \rangle)$

$$S_{\text{MC}}[a] = \sum_{i \geq 1} \frac{1}{(i+1)!} \langle a, \mu_i(a, \dots, a) \rangle$$

Chain complex (μ_1)



Yang–Mills theory in the BV formalism

- Other non-vanishing higher products

$$
\begin{aligned}
[\mu_2(A,c)]^a &= g f^a_{bc} c^b c^c \ , \quad [\mu_2(A,c)]^a_\mu = -g f^a_{bc} A^b_\mu c^c \\
[\mu_2(A^+,c)]^a_\mu &= -g f^a_{bc} A^{+b}_\mu c^c \ , \quad [\mu_2(c,c^+)]^a = g f^a_{bc} c^b c^{+c} \\
[\mu_2(A,A)]^a_\mu &= -3!\kappa f^a_{bc} \partial^\nu (A^b_\nu A^c_\mu) \\
[\mu_2(A,A^+)]^a &= 2g f_{bc}{}^a \Big(\partial^\nu (A^b_\nu A^c_\mu) + 2A^{b\nu} \partial_{[\nu} A^c_{\mu]} \Big) \\
[\mu_3(A,A,A)]^a_\mu &= 3! g^2 f_{ed}{}^b f_{bc}{}^a A^{\nu c} A^d_\nu A^e_\mu
\end{aligned}
$$

- Cyclic structure

$$
\begin{aligned}
\langle A, A^+ \rangle &= \int_{\mathbb{M}^d} \mathrm{d}^d x \, A^a_\mu A^{+a\mu} \ , \quad \langle b, b^+ \rangle = \int_{\mathbb{M}^d} \mathrm{d}^d x \, b^a b^{+a} \ , \\
\langle c, c^+ \rangle &= \int_{\mathbb{M}^d} \mathrm{d}^d x \, c^a c^{+a} \ , \quad \langle \bar{c}, \bar{c}^+ \rangle = -\int_{\mathbb{M}^d} \mathrm{d}^d x \, \bar{c}^a \bar{c}^{+a}
\end{aligned}
$$

Yang–Mills theory in the BV formalism

- Gauge-fixing: gauge-fixing fermion

$$\Psi = - \int \mathrm{d}^d x \, \bar{c}_a \big(\partial^\mu A^a_\mu + \tfrac{\xi}{2} b^a \big) \, .$$

with ξ real parameter

- Gauge-fixed action

$$S^{\mathrm{gf}}_{\mathrm{YM}} = \int \mathrm{d}^d x \left\{ - \tfrac{1}{4} F_{a\mu\nu} F^{a\mu\nu} - \bar{c}_a \partial^\mu (\nabla_\mu c)^a + \tfrac{\xi}{2} b_a b^a + b_a \partial^\mu A^a_\mu \right\}$$

$$S^{\mathrm{gf}}_{\mathrm{YM}} = \int \mathrm{d}^d x \left\{ \tfrac{1}{2} A_{a\mu} \Box A^{a\mu} + \tfrac{1}{2} (\partial^\mu A^a_\mu)^2 - \bar{c}_a \Box c^a + \tfrac{\xi}{2} b_a b^a + b_a \partial^\mu A^a_\mu \right\} + S^{\mathrm{int}}_{\mathrm{YM}}$$

Structural theorems for L_∞-algebras

Definitions

We call an L_∞-algebra

- *minimal*, if $\mu_1 = 0$
- *strict*, if $\mu_i = 0$ for $i > 2$, i.e. if it is a differential graded Lie algebra

Theorems

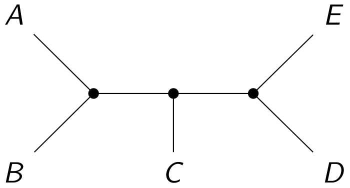
- *Minimal model theorem*. Every (cyclic) L_∞-algebra is quasi-isomorphic to a minimal (cyclic) L_∞-algebra
- *Strictification theorem*. Every (cyclic) L_∞-algebra is quasi-isomorphic to a strict (cyclic) L_∞-algebra

Analogous definitions and results apply to (cyclic) A_∞- and C_∞-algebras

Strict L_∞-algebras

- Every perturbative Lagrangian field theory is equivalent to a theory with only cubic interactions (*strictification*)

$$\mathrm{f}_{ab}{}^{f}\mathrm{f}_{fcg}\mathrm{f}_{de}{}^{g}A^aB^bC^cD^dE^e \quad \Longleftrightarrow$$



- A quintic interaction term $\mathrm{f}_{ab}{}^{f}\mathrm{f}_{fcg}\mathrm{f}_{de}{}^{g}A^aB^bC^cD^dE^e$ can be strictified inserting auxiliary fields, and it is equivalent to

$$\bar{G}_{1a}G_1^a + \bar{G}_{2a}G_2^a + \mathrm{f}_{ab}{}^{f}A^aB^b\bar{G}_{1f} + \mathrm{f}_{fcg}G_1^fC^cG_2^g + \mathrm{f}_{de}{}^{g}\bar{G}_{2g}D^dE^e$$

$BV^{■}$-algebras and CK duality

- Yang–Mills action can be cast in a cubic, CK-dual form ☞ Leron's talk
- Consider the associated strict L_∞-algebra $\mathfrak{L}^{\text{st}}$: it factorize into the gauge algebra $\mathfrak{g}$ and a strict C_∞-algebra $\mathfrak{C}^{\text{st}}$

$$\mathfrak{L}^{\text{st}} = \mathfrak{g} \otimes \mathfrak{C}^{\text{st}}$$

- $\mathfrak{C}^{\text{st}}$ contains binary bracket m_2 of degree 0

$$m_2 : \text{field} \times \text{field} \to \text{antifield}$$

- Kinematic algebra: we want to define a new bracket of degree -1: extend $\mathfrak{C}^{\text{st}}$ to a $BV^{■}$-algebra

$BV^{\blacksquare}$-algebras and CK duality

A graded vector space $\mathfrak{B}$, equipped with

- Hodge triple $(d, h, \blacksquare)$

$$\text{fields} \underset{h}{\overset{d}{\rightleftarrows}} \text{antifields}$$

$$d^2 = 0, \quad h^2 = 0, \quad dh + hd = \blacksquare$$

- graded commutative product m_2 of degree 0
- a new product $\{-,-\}$ of degree -1

$$\{v_1, v_2\} = hm_2(v_1, v_2) - m_2(h(v_1), v_2) - (-1)^{|v_1|} m_2(v_1, h(v_2))$$

- and more technical stuff

$BV^{\blacksquare}$-algebras and CK duality

A strictified gauge field theory with L_∞-algebra $\mathfrak{L}^{\text{st}}$ is manifestly CK-dual if and only if there is a factorisation $\mathfrak{L}^{\text{st}} = \mathfrak{g} \otimes \mathfrak{C}^{\text{st}}$ such that $\mathfrak{C}^{\text{st}}$ can be enhanced to a $BV^{\blacksquare}$-algebra.

Double copy

YM theory: $\mathfrak{L}^{\text{st}} = \mathfrak{g} \otimes \mathfrak{B}$. Naively: we want to replace $\mathfrak{g}$ with a copy of $\mathfrak{B}$

- In the factorization $\mathfrak{g} \otimes \mathfrak{C}$, $\mathfrak{g}$ can be thought as a $BV^{\blacksquare}$-algebra
- Tensor product between associative, commutative and Lie algebras

$\otimes$	Ass	Com	Lie
Ass	Ass	Ass	–
Com	Ass	Com	Lie
Lie	–	Lie	–

- For two compatible algebras $\mathfrak{A}, \mathfrak{B}$

$$m_2^{\mathfrak{A}\otimes\mathfrak{B}}(a_1 \otimes b_1, a_2 \otimes b_2) \;=\; m_2^{\mathfrak{A}}(a_1, a_2) \otimes m_2^{\mathfrak{B}}(b_1, b_2)$$

- $BV^{\blacksquare}$-algebras contain differential graded Lie algebras: we should be careful when we define tensor product!

Double copy

- There is a notion of tensor product between two $BV^{\blacksquare}$-algebras

$$\mathfrak{L} \subset \mathfrak{B}_1 \otimes \mathfrak{B}_2$$

- Inspired by string theory: $\mathcal{H}_{\text{closed}} = \mathcal{H}_{\text{open}} \otimes \mathcal{H}_{\text{open}}$
- Section condition: states $|p, \ldots\rangle \otimes |p, \ldots\rangle$
- Level matching: $(b_0 - \tilde{b}_0)\,|\psi\rangle = 0$, $(L_0 - \tilde{L}_0)\,|\psi\rangle = 0$
- String theory: Hodge triple $d = Q_{\text{BRST}}$, $h = b_0$, $\blacksquare = L_0$

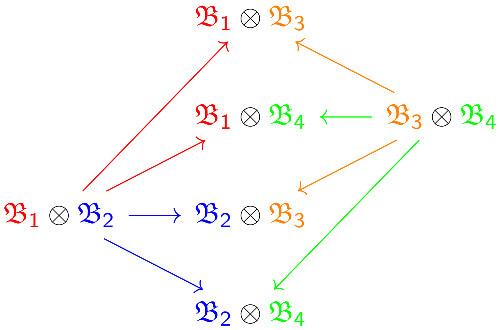
Double copy

- Field theory: Hodge triple $d = m_1$, $h = [1]$, $\blacksquare = \square$
- Tensor product $\mathfrak{B}_1 \otimes \mathfrak{B}_2$: implementation of section condition and level matching
- Double copy is readily interpreted in terms of tensor product of $BV^{\blacksquare}$-algebras

Double copy

- A strictified field theory with L_∞-algebra $\mathfrak{L}^{\text{st}}$ is manifestly CK-dual if and only if it factorizes into a tensor product of $BV^{\blacksquare}$-algebras

- Given two $BV^{\blacksquare}$-algebras, tensor product combines them into a differential graded algebra: *syngamy* ☞ Leron's talk



Outlook

- Mathematical formulation of kinematic algebra, CK duality and double copy in terms of homotopy algebras
- Lagrangian incarnation of CK duality and double copy
- Casting an action into CK-dual form [☞ Leron's talk]: homotopy algebra interpretation
- Generalization to the non-strict case: $BV_\infty^{■}$-algebras
- Computational efficiency?
- Link to string theory

Thank you for listening!